Schrodinger's Wave Equation.

In 1926, Schrodinger's using de-Broglie's idea of matter waves & developed a mathematical formula which is known as wave-mechanics derivation.

Let us consider the vibration of a stretched string.

If 'ω' be the amplitude of any point whose co-ordinate is 'x' at time 't'

The appropriate form of the wave equation may be written as follows

$$\frac{\partial^2 \omega}{\partial x^2} = \frac{1}{u^2} \frac{\partial^2 \omega}{\partial t^2} \quad \text{——— (i)}$$

Where 'u' is the velocity of propagation of the wave on separating the variables, this differential equation may be written as

$$\omega = f(x) \cdot g(t) \quad \text{——— (ii)}$$

Where $f(x)$ is a function of the co-ordinate 'x' only and $g(t)$ is a function of time 't' only.

For the motion of standing waves such as occurring in a stretched string, it is possible to express $g(t)$ as

$$g(t) = A \sin \omega t = A \sin 2\pi \nu t \quad \text{— (iii)}$$

where ν is the vibrational frequency & A is constant and it stands for maximum amplitude.

From eq^n (ii) & (iii)

$$\omega = f(x) \cdot A \sin 2\pi \nu t$$

On differentiating the above equation with respect to 't'

we have,

$$\frac{\partial \omega}{\partial t} = f(x) \cdot A \cos 2\pi \nu t \cdot 2\pi \nu$$

$$\frac{\partial^2 \omega}{\partial t^2} = f(x) (-) A \sin 2\pi \nu t \cdot 2\pi \nu \cdot 2\pi \nu$$

or

$$\frac{\partial^2 \omega}{\partial t^2} = -4\pi^2 \nu^2 f(x) \cdot A \sin 2\pi \nu t \qquad [\because A \sin 2\pi \nu t = g(t)]$$

$$= -4\pi^2 \nu^2 f(x)\, g(t) \quad \text{——— (iv)}$$

Now differentiating the eq^n (ii) with respect to x

$$\omega = f(x) \cdot g(t)$$

$$\frac{\partial \omega}{\partial x} = \frac{\partial f(x)}{\partial x} \cdot g(t)$$

or

$$\frac{\partial^2 \omega}{\partial x^2} = \frac{\partial^2 f(x)}{\partial x^2} \cdot g(t) \quad \text{——— (v)}$$

From equation (iv) and (v) i.e substituting the value of $\frac{\partial^2 \omega}{\partial t^2}$ and $\frac{\partial^2 \omega}{\partial x^2}$, in equation (i) i.e $\frac{\partial^2 \omega}{\partial x^2} = \frac{1}{u^2} \cdot \frac{\partial^2 \omega}{\partial t^2}$

$$\frac{\partial^2 f(x)}{\partial x^2} \cdot g(t) = \frac{1}{u^2} \left(-4\pi^2 \nu^2\right) f(x) \cdot g(t)$$

$$\text{or} \quad \frac{\partial^2 f(x)}{\partial x^2} = -\frac{4\pi^2 \nu^2}{u^2} f(x) \text{ --- (vi)}$$

But ν and u are related by the equation

$u = \nu\lambda$ or $u^2 = \nu^2\lambda^2$, on Putting this value in the above eqn we have.

$$\therefore \frac{\partial^2 f(x)}{\partial x^2} = -\frac{4\pi^2 \cancel{\nu^2}}{\cancel{\nu^2}\lambda^2} f(x) \text{ ---}$$

$$\frac{\partial^2 f(x)}{\partial x^2} = -\frac{4\pi^2}{\lambda^2} f(x) \text{ --- (vii)}$$

It is the expression for the wave eqn in one direction and it can be extended in three directions, expressed by the Co-ordinate x, y and z. If $f(x)$ for one Co-ordinate is replaced by three Co-ordinate x, y & z i.e $\psi(x, y z)$, which is amplitude function for three Co-ordinates. then eqn (vii) take the form as follows.

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} = -\frac{4\pi^2}{\lambda^2} \psi \text{ --- (viii)}$$

using the Symbol ∇^2 for differential Operator

i.e $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

Here ∇^2 (Del squared) is known as Laplacian operator.

then eqn (viii) may be replaced by

$$\nabla^2 \psi = -\frac{4\pi^2}{\lambda^2} \psi \text{ --- (ix)}$$

The above treatment is applicable to all particles including electrons. atoms and Photons.

By using, de-Broglie's relation $\lambda = \frac{h}{mu}$ in to eqn (ix) we have

$$\nabla^2 \psi = -\frac{4\pi^2}{h^2} m^2 u^2 \psi \text{ --- (x)}$$

where m is mass of Particle, u is velocity and h is Plank Constant.

be Continued . . . .

But we know that the total Energy of a Particle is the sum of its Potential energy and Kinetic energy

If, E = Total energy, U = Potential energy and Kinetic energy = $\frac{1}{2}mu^2$

$\therefore$ $E = K.E + P.E$

$E = \frac{1}{2}mu^2 + U$

or $2(E-U) = mu^2$ ——— (xi)

On Substituting the value of mu^2 in eqn (x)

we have $\nabla^2\psi = -\frac{4\pi^2 m}{h^2} \cdot 2(E-U) \cdot \psi$

or $\nabla^2\psi = -\frac{8\pi^2 m}{h^2}(E-U)\psi$

or $\nabla^2\psi + \frac{8\pi^2 m}{h^2}(E-U)\psi = 0$ ——— (xii)

It is the required form of Schrodinger's Wave Equation.